The Ninth Annual Herzog Prize Examination

November 7, 1981

Problem 1: Does the inequality

$$(2n)^n + (2n+1)^n \geq (2n+2)^n$$

hold for all positive integers n?

Problem 2: (L.M. Kelly) A point P moves on the positive x-axis and point Q on the positive y-axis so that the line PQ is always tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

Prove that when the triangle POQ has minimum area the point of tangency will bisect the segment PQ. (Here O is the origin.)

Problem 3: (J. Marik) Let f, g be nondecreasing on $[0, 1]$. Prove that

$$\int_0^1 f(x)\,dx \int_0^1 g(x)\,dx \leq \int_0^1 f(x)g(x)\,dx .$$

Problem 4: (L.M. Kelly) If a, b, c are real and $b^2 < 2ac$, prove that the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

cannot have all real roots.

Problem 5: Prove

$$\sin 10 \sin 30 \sin 50 \sin 70 = \frac{1}{16} .$$

Problem 6: Three points are taken at random on a unit sphere. What is the probability that the area of the spherical triangle exceeds the area of a great circle?